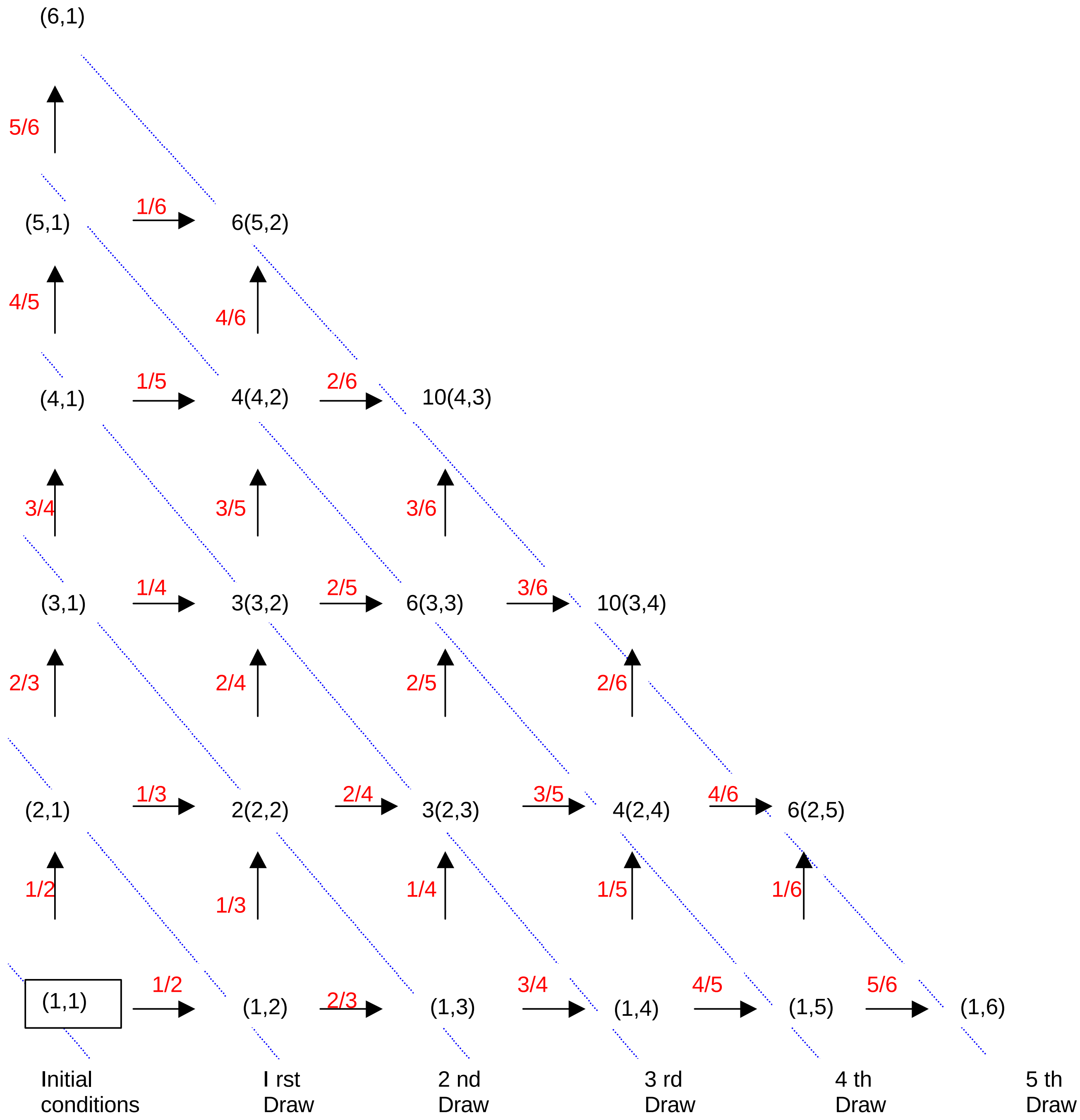
1 - Blackwell urn with balls of two colours.

1 - Initial conditions : An urn contains one white ball and one black ball.

2 - Rule : One holds a draw, if the white ball has been drawn, one puts back the white ball plus another white ball in the urn ; if the black ball has been drawn, one puts back the black ball plus another black ball in the urn.

3 - One repeats a large number of times this rule.

Let us show the tree of the events (**figure 1**):



Let us denote :

1 – A discrete two-dimensional vector space E on N, (N is the set of natural numbers),

2 - The vector $v = (w,b), v \in \mathrm{E}$ corresponds to an urn, which includes w white balls and b black balls

3 - $V_p = \{v_{p,1}, v_{p,2}, ..., v_{p,k}\}$ the set vectors corresponding at all the possible urns after p draws, The initial condition correspond to the vector $v_0 = (1,1), V_0 = \{v_0\}$,

After p draws, the state of the urn is the vector $v_p = v_0 + w(0,1) + b(1,0) = (w+1, b+1)$ and $w + b = p$.

If $v_{p,i} = (w,b)$, the probability to draw a white ball is $\wp(w) = \dfrac{w}{w+b}$ and the next state of the urn is $v_{p+1,j} = (w,b) + (1,0) = (w+1,b)$, the probability to draw a black ball is $\wp(b) = \dfrac{b}{w+b}$ and the next state of the urn is $v_{p+1,i} = (w,b) + (0,1) = (w,b+1)$.

4 – From each vector (**figure 1**)(i.e. (w,b)), start two arrows :

- a - To the horizontal arrow correspond :
  - i - the event "to draw a black ball",
  - ii - the addition of the vector $(0,1)$ to (w,b) and
  - iii - the probability $\wp(w)$,
- b – To the vertical arrow correspond :
  - i - the event "to draw a white ball",
  - ii - the addition of the vector $(1,0)$ to (w,b) and
  - iii - the probability $\wp(b)$.

<u>Theorem I</u> : the number of possible distinct urns after p draws is $|V_p| = C(p+2-1, p) = p+1$.

At the p-th draw, $v_p = v_0 + w(0,1) + b(1,0) = (w+1, b+1)$ and $w + b = p$.

The number of ways to choose p objects from a collection of two objects (the vectors $(1,0)$ and $(0,1)$) is the number of distinct solutions of the equation $w + b = p$, w and b being natural numbers, this number is the combinations with repetitions $C(p+2-1, p)$:

$$C(p+2-1, p) = \frac{(p+1)!}{1!\,p!} = p+1$$

<u>Remark</u> : $C(p+2-1, p)$ is the coefficient of x^p in the generating function :

$$(1 + x + x^2 + x^3 + x^4 + ...) \times (1 + x + x^2 + x^3 + x^4 + ...) = (\frac{1}{1-x})^2 =$$
$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + ... + (p+1)x^p + ...$$

<u>Theorem II</u> : the number of distinct ways from initial vector $v_0=(1,1)$ to the vector $v_p=(w+1,b+1)$ is $\dfrac{(w+b)!}{w!\,b!}$.

As $(w+1,b+1)=(1,1)+w(0,1)+b(1,0)$, the number of different ways to go from the vector $v_0=(1,1)$ to the vector $v_p=(w+1,b+1)$ is equal to the number of collections of p objects arranged in order from the repetition w times of the objects $(1,0)$ and the repetition b times of the object $(0,1)$. This number is the permutations with duplicates $\dfrac{p!}{w!\,b!}=\dfrac{(w+b)!}{w!\,b!}$.

These numbers are written in front of each vector on **figure 1**.

<u>Theorem III</u> : the probability to reach $v_p=(w+1,b+1)$ from $v_0=(1,1)$ is $\dfrac{w!\,b!}{(w+b+1)!}$.

$$(w,b+1)\xrightarrow{\wp(1)}(w+1,b+1)\,,\ \wp(1)=\frac{w}{w+b+1}\,,$$

$$(w-1,b+1)\xrightarrow{\wp(2)}(w,b+1)\,,\ \wp(2)=\frac{w-1}{w+b}\,,$$

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$$(1,b+1)\xrightarrow{\wp(w)}(2,b+1)\,,\ \wp(w)=\frac{1}{b+2}\,,$$

$$(1,b)\xrightarrow{\wp(w+1)}(1,b+1)\,,\ \wp(w+1)=\frac{b}{b+1}\,,$$

$$(1,b-1)\xrightarrow{\wp(w+2)}(1,b)\,,\ \wp(w+2)=\frac{b-1}{b}\,,$$

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$$(1,1)\xrightarrow{\wp(w+b)}(1,2)\,,\ \wp(w+b)=\frac{1}{2}$$

All these events beings independent,

$$(1,1)\xrightarrow{\wp}(w+1,b+1)\,,\ \wp=\wp(1).\wp(2)....\wp(w).\wp(w+1).\wp(w+2)....\wp(w+b).$$

$$\wp=\frac{w}{w+b+1}.\frac{w-1}{w+b}.....\frac{1}{b+2}.\frac{b}{b+1}.\frac{b-1}{b}.....\frac{1}{2}=\frac{w!\,b!}{(w+b+1)!}\,.$$

<u>Theorem IV</u> : the probabilities of the $p+1$ events $v_{p,i}\in V_p$ are $\dfrac{1}{p+1}$.

The $p+1$ events $v_{p,i}\in V_p$ obtained after p draws, can be written $(w+1,b+1)$ with $w+b=p$.

The probability of the event $v_{p,i}$ is the product of the number of occurrences $\frac{(w+b)!}{w!b!}$ of this event (given by the theorem II) by the probability $\frac{w!b!}{(w+b+1)!}$ to reach $v_p = (w+1, b+1)$ from $v_0 = (1,1)$ (given by the theorem III) :

$$\frac{(w+b)!}{w!b!}\cdot\frac{w!b!}{(w+b+1)!} = \frac{1}{w+b+1} = \frac{1}{p+1}$$

2 - GENERALISATION : Blackwell urn with balls of k different colours.

1 - Initial conditions : An urn contains k balls of different colours.

2 - Rule : One holds a draw, each time a ball of one colour is drawn, one replaces 2 balls of the same colour in the urn.

3 - One repeats a large number of times this rule.

Let us denote :

1 – A discrete k dimensional vector space E on N, (N is the set of natural numbers),
2 – The vector $v=(x_1,x_2,...,x_k), v\in \mathrm{E}$ corresponds to an urn which includes x_1 balls of the colour #1, x_2 balls of colour #2,…, x_k balls of the colour #k,
3 - $V_p = \{v_{p,1}, v_{p,2}, ..., v_{p,k}\}$ the set vectors corresponding at all the possible urns after p draws, The initial condition correspond to the vector $v_0 = (1,1,...,1), V_0 = \{v_0\}$.

Theorem I bis : the number of possible distinct urns after p draws is $|V_p|=C(k+p-1,p)$.

The number of ways to choose p objects from a collection of k objects is the number of distinct solutions of the equation $x_1+x_2+...+x_k=p$, this number is the combinations with repetitions $C(k+p-1,p)$:

$$C(k+p-1,p)=\frac{(k+p-1)!}{(k-1)!p!}$$

Remark : $C(k+p-1,p)$ is the coefficient of y_p in the generating function $(\frac{1}{1-y})_k$.

Example : For k=3,

$$(\frac{1}{1-y})_3=1+C(3,1)y+C(4,2)x_{y2}+C(5,3)y_3+C(6,4)y_4+C(7,5)y_5+...$$

$$=1+3y+6y_2+15y_3+21y_4+28y_5+...$$

Theorem II bis : the number of distinct ways from the initial vector $v_0=(1,1,..,1)$ to the vector $v_p=(x_1+1,x_2+1,...,x_k+1)$ is $\frac{p!}{x_1!x_2!...x_k!}$.

The number of different ways to go from the initial vector to the vector $v_p=(x_1+1,x_2+1,...,x_k+1)$ is equal to the number of collections of p objects arranged in order from the repetition x_1 times of the objects of colour #1 and the repetition x_2 times of the object of colour #2, …, and the repetition x_k times of the object of colour #k. This number is the permutations with duplicates: $\frac{(x+x_2+..+x_k)!}{x_1!x_2!\dots x_k!}=\frac{p!}{x_1!x_2!\dots x_k!}$.

<u>Theorem III bis</u> : the probability to reach $v_p=(x_1+1,x_2+1,\dots,x_k+1)$ from the initial vector $v_0=(1,1,..,1)$ is $\wp=\frac{x_1!x_2!\dots x_k!(k-1)!}{(x_1+x_2+\dots+x_k+k-1)!}$.

$$(x_1,x_2+1,\dots+x_k+1)\xrightarrow{\wp(1)}(x_1+1,x_2+1,\dots,x_k+1)\ ,\ \wp(1)=\frac{x_1}{x_1+x_2+\dots+x_k+k-1}\ ,$$

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$$(1,x_2+1,\dots+x_k+1)\xrightarrow{\wp(x)}(2,x_2+1,\dots,x_k+1)\ ,\ \wp(x_1)=\frac{1}{x_2+\dots+x_k+k}\ ,$$

$$(1,x_2,\dots+x_k+1)\xrightarrow{\wp(x_1+1)}(1,x_2+1,\dots,x_k+1)\ ,\ \wp(x_1+1)=\frac{x_2}{x_2+\dots+x_k+k-1}\ ,$$

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$$(1,1,\dots+l+1)\xrightarrow{\wp(x_1+x_2)}(1,2,\dots,l+1)\ ,\wp(x_1+x_2)=\frac{1}{x_3+\dots+x_k+k}\ ,$$

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$$(1,1,\dots+l)\xrightarrow{\wp(x_1+x_2+\dots+)}(1,2,\dots,l+1)\ ,\wp(x_1+x_2+\dots+)=\frac{l}{x_k+k-1}\ ,$$

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$$(1,1,\dots,2)\xrightarrow{\wp(x_1+x_2+\dots+x_k)}(1,1,\dots,2)\ ,\ \wp(x_1+x_2+\dots+x_k)=\frac{1}{k}\ ,$$

$$\wp=\wp(1)\dots\dots\wp(x_1)\dots\dots\wp(x_1+1)\dots\dots\wp(x_1+x_2)\dots.\wp(x_1+x_2+\dots+x_k)\ ,$$

$$\wp=\frac{x_1}{x_1+x_2+\dots+x_k+k-1}\dots..\frac{1}{x_2+\dots+x_k+k}.\frac{x_2}{x_2+\dots+l+k-1}\dots..\frac{1}{x_3+\dots+x_k+k}\dots..\frac{l}{x_k+k-1}\dots\frac{1}{k}\ ,$$

$$\wp=\frac{x_1}{x_1+x_2+\dots+x_k+k-1}\dots..\frac{1}{x_2+\dots+x_k+k}.\frac{x_2}{x_2+\dots+x_k+k-1}\dots..\frac{1}{x_3+\dots+x_k+k}\dots..\frac{x_k}{x_k+k-1}\dots\frac{1}{k}\ ,$$

$$\wp=\frac{x_1!x_2!\dots x_k!}{(x_1+x_2+\dots+x_k+k-1)\dots(x_2+\dots+x_k+k)(x_2+\dots+x_k+k-1)\dots(x_3+\dots+x_k+k)\dots(x_k+k-1)\dots..k}\ ,$$

$$\wp=\frac{x_1!x_2!\dots x_k!(k-1)!}{(x_1+x_2+\dots+x_k+k-1)!}\ .$$

<u>Theorem IV bis</u> : the probabilities of the $|V_p|=C(k+p-1,p)$ events $v_{p,i}\in V_p$ are $\frac{1}{C(k+p-1,p)}$.

The $C(k+p-1,p)$ events $v_{p,i}\in V_p$ obtained after p draws, can be written $v_{p,i}=(x_1+1,x_2+1,\dots,x_k+1)$ with $x_1+x_2+\dots+x_k=p$.

The probability of the event $v_{p,i}$ is the product of the number of occurrences $\frac{p!}{x_1!x_2!\ldots x_k!}$ of this event (given by the theorem II bis) by the probability $\frac{x_1!x_2!\ldots x_k!(k-1)!}{(x_1+x_2+\ldots+x_k+k-1)!}$ to reach $v_p=(x_1+1,x_2+1,\ldots,x_k+1)$ from $v_0=(1,1,\ldots,1)$ (given by the theorem III bis) :

$$\frac{p!}{x_1!x_2!\ldots x_k!}\cdot\frac{x_1!x_2\ldots x_k!(k-1)!}{(k+p-1)!}=\frac{p!(k-1)!}{(k+p-1)!}=\frac{1}{C(k+p-1,p)}$$

CONCLUSION :

The $C(k+p-1,p)$ events $v_{p,i}\in V_p$ obtained after p draws have the same probability $\frac{1}{C(k+p-1,p)}$

3 – Simulations.

- 32,000 simulations have been made with p=32000 and k=2,3,4 and 5.
- At the end of each simulation, the numbers $x_1+1,x_2+1,\ldots,x_k+1$ of balls of colours #1,#2,…,#k of the urn, obtained after p draws are written down.
- At the end of 32,000 simulations, bar charts for each colours are plotted :

  - X-axis corresponds to the different values of x_n+1 corresponding to the colour n ($n\in[1,\ldots,k]$).
  - Y-Axis corresponds the frequency z_n of the values x_n+1.

- In order to understand these results, the theoretical value of $z_n=F(p,k,n)$ must be given. :

After p draws :

- The number of balls in the urn is $k+p=x_1+1+x_2+1+\ldots+x_k+1$
$$=x_1+x_2+\ldots+x_k+p$$

- The number of possible distinct urns is $|V_p|=C(k+p-1,p)=\frac{(k+p-1)!}{(k-1)!p!}$.
- The number z_n of possible distinct urns containing x_n+1 balls of colour n corresponds to the number of way to choose k+p-n objects from a collection k-1 objects :

$$z_z=\frac{(k+p-x_n-2)!}{(k-2)!(p-x_n)!}$$

If the ball of colour n has never been drawn, $x_n+1=0$,
If the ball of colour n has been drawn at each draw, $x_n+1=p$.

Therefore $x_n\in[1,p+1]$

- For $k=3$, $z_z=\frac{(p-x_n+1)!}{(p-x_n)!}=p-x_n+1$.
- For $k=4$, $z_z=\frac{(p-x_n+2)!}{2!(p-x_n)!}=\frac{1}{2}(p-x_n+2)(p-x_n+1)$

- For $k=5$, $z_z=\frac{(p-x_n+3)!}{3!(p-x_n)!}=\frac{1}{2}(p-x_n+3)(p-x_n+2)(p-x_n+1)$
- In a general way, z_n is a polynomial of degree k-2 in p.

4 – Another rules.

<u>1 - Initial conditions</u> : An urn contains one white ball and one black ball.

<u>2 - Rule</u> : One holds a draw, if the white ball has been drawn, one puts back the white ball plus a black ball in the urn ; if the black ball has been drawn, one puts back the black ball plus a fixed number l of white balls in the urn.

3 - One repeats a large number of times this rule.

<u>Problem.</u>

- An urn contains b black balls and $w=k.b$ white balls, $\frac{b}{w}=\frac{1}{k}$

- The rule 2 (above) is applied.

- What must be the value of l in order the ratio $\frac{E(b)}{E(w)}=\frac{1}{k}$?

$E(b)$ and $E(w)$ are the expectations of b and w.

$$E(b)=\wp(w).(b+1)+\wp(b).b \qquad (1)$$
$$E(b)=\wp(b).b+\wp(n).(b+l) \qquad (2)$$

$\wp(w)=\frac{k.b}{b+k.b}=\frac{k}{k+1}$ and $\wp(b)=\frac{b}{b+k.b}=\frac{1}{k+1}$ therefore (1) and (2) becomes :

$$E(b)=\frac{k}{k+1}.(b+1)+\frac{1}{k+1}.b=b+\frac{k}{k+1}$$
$$E(w)=\frac{k}{k+1}.w+\frac{1}{k+1}.(w+l)=w+\frac{l}{k+1}=k.b+\frac{l}{k+1}$$

$$\frac{E(b)}{E(w)}=\frac{n+\frac{k}{k+1}}{k.n+\frac{l}{k+1}} \quad \text{if } l=k^2 \quad \frac{E(b)}{E(w)}=\frac{b+\frac{k}{k+1}}{k.b+\frac{k^2}{k+1}}=\frac{1}{k}$$

Consequently, after a large number of draws, the ratio b/w has $1/\sqrt{l}$ for limit. In peculiar if $l=1$, $b/w=1$